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Consensus of Multi-Agent Systems with Additive Delays

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(Abstract) In this paper, we consider the problem of consensus for continuous-time multi-agent systems with additive delays. Such a problem receives little attention in recent years. By transforming the multi-agent system into a reduced-order system with distributed delays, we establish a sufficient condition to consensus of the system if the involved digraph has a spanning tree. Our result not only gives a tolerable bound of delays, but also complement the analysis for consensus of multi-agent systems with delays.

Keywords: Multi-agent System; Additive Delay; Reduced-order System; Distributed Delay.

1. INTRODUCTION

Recently, more and more researchers have focused their attention on distributed coordination of multi-agent systems. Its applications can be found in cooperative control of unmanned air vehicles, formation control of mobile robots, flocking of social insects, etc. A critical problem for coordinated control is to design appropriate control strategies or protocols based on local information such that the group of dynamic agents can reach an agreement on certain quantities of interest. Such problem is usually called consensus.

Consensus problems have been extensively studied (see the review papers [1] and [2], and references therein). A theoretical framework for consensus problems of continuous-time multi-agent systems was presented in [3], where average consensus in directed networks with fixed and switching topologies as well as uniform constant communication delay were studied respectively. Ren et al. (2005) extended the results in [4] to the case of directed networks with switching interaction topologies under a milder condition. We are here concerned with consensus problem in the existence of communication delays.

So far, two types of consensus protocols with delays have been considered. One is both the state of each agent and its neighbors are affected by symmetric delays. Another is that delays only affect the state received from neighbors of each agent (asymmetric case). For the first case, Saber (2005) studied consensus problem in undirected networks with fixed topology and common constant delay, and established a necessary and sufficient consensus condition in [3]. Bliman et al. (2005) extended the result of Saber to the case of nonuniform constant and time-varying delays in [5]. For the second case, Moreau (2004) studied consensus in directed networks with dynamically changing topology and common constant delay in [6]. By set-value Lyapunov theory, the results of Moreau were extended to the case of time-varying delays in [7]. Based on a linear matrix inequality method, consensus for directed networks with non-uniform time-varying delays was investigated in [8-14].

In this paper, we will study consensus of multi-agent systems with additive delays, where one kind of delay are internal delays of agent, another are external delays (i.e., the delay of receiving information from neighbors). To the best of our knowledge, little has been known about consensus for this case.

Different from the above two cases of delay, it leads to difficulties due to the existence of additive delays. It seems to us that most of existing methods can not be applied to this case directly. When there exists at least one agent whose dynamic equation is free of delay (e.g., a leader with dynamic equation $\dot{x}_i = 0$), we analyze consensus of the multi-agent system with two additive delays by introducing a reduced-order transformation.

Throughout this paper, A^T means the transpose of the matrix A. We say $X > Y$ if $X - Y$ is positive definite, where X and Y are symmetric matrices of same dimensions. I_p is a $p \times p$ dimensional identity matrix. Denote $\tilde{a} = [a, a, \cdots, a]^T$ by a column vector of appropriate dimension, where a is a constant.

2. PROBLEM STATEMENT

A weighted digraph is denoted by $G = (V, E, A)$, $V = \{1, 2, \cdots, n\}$ is the set of nodes with $n \geq 2$, $E \subseteq V \times V$ is the set of edges, $A = [a_{ij}]$ is an $n \times n$ dimensional weighted adjacency matrix with $a_{ii} = 0$. A directed edge of G is denoted by $e_{ij} = (i, j)$, $e_{ij} \in E$ if and only if $a_{ji} > 0$. The Laplacian matrix $L(G) = [l_{ij}]_{n \times n}$ of digraph G is defined by $l_{ii} = \sum_{k=1}^{n} a_{ik}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. It is easy to see that $L(G)$ has at least one zero eigenvalue and $L(G)\tilde{1} = 0$. When $e_{ij}\in E$, we mean the jth agent receives information

from the ith agent, and agent i is the neighbor of agent j.

If (i, j) is an edge of G, node i is called the parent of node j. A directed tree is a directed graph, where every node, except one special node without any parent, which is called the root, has exactly one parent, and the root can be connected to any other nodes through paths. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph.

Consider the following multi-agent system with additive delays

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[x_j(t-\tau_i-\sigma_{ij}) - x_i(t-\tau_i)], \tag{1}$$

where $t \geq 0$, $\tau_i \geq 0$ $(i \in V)$ denotes the internal delay of agent i, and $\sigma_{ij} \geq 0$ denotes the external delay, i.e., the delay of receiving its neighbors' information. Unlike most of multi-agent systems with delays in the literature, system (1) contains additive delays τ_i and σ_{ij}.

Say system (1) asymptotically solves consensus, if $\lim_{t\to\infty}[x_i(t) - x_j(t)] = 0$ for $i \neq j$ and $i, j \in V$.

In the following, we assume that there exists at least one agent whose dynamic equation is free of delay. For example, there is a leader among agents with dynamic equation $\dot{x}_i = 0$. Without loss of generality, suppose that

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)], \tag{2}$$

where $i = 1, 2, \cdots, p$ with $n > p \geq 1$. For the sake of convenience, we also assume that $\tau_i = \tau > 0$ and $\sigma_{ij} = \sigma > 0$ for $i = p+1, \cdots, n$. The case of multiple delays can be similarly discussed based on the following analysis procedure. Hence, agent i for $p+1 \leq i \leq n$ satisfies the following dynamic equation

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[x_j(t-\tau-\sigma) - x_i(t-\tau)]. \tag{3}$$

3. MAIN RESULTS

Before giving the main result of this paper, we first introduce the following lemmas which play a key role in the proof.

Lemma 1 [10]. The digraph $G = (V, E, A)$ has a spanning tree if and only if the matrix $EL(G)F$ is Hurwitz stable, where $E = \begin{bmatrix} \tilde{1} & -I_{n-1} \end{bmatrix}$ and $F^T = \begin{bmatrix} \tilde{0} & I_{n-1} \end{bmatrix}$.

Rewrite (3) as follows:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[x_j(t-\eta) - x_i(t-\eta)] + l_{ii}[x_i(t-\eta) - x_i(t-\tau)],$$

where $\eta = \tau + \sigma$, $i = p+1, \cdots, n$, and l_{ii} is the diagonal element of $L(G)$. Thus, the multi-agent system satisfying (2) and (3) takes the following matrix form

$$\dot{x}(t) = -L_1 x(t) - L_2 x(t-\eta) + D[x(t-\eta) - x(t-\tau)], \tag{4}$$

where $x^T = [x_1, x_2, \cdots, x_n]$,

$$L_1 = diag\{I_p, O_1\}L(G),$$
$$L_2 = diag\{O_2, I_{n-p}\}L(G),$$
$$D = diag\{0, \cdots, 0, l_{p+1\,p+1}, \cdots, l_{nn}\},$$

O_1 and O_2 are zero matrices of appropriate dimensions. It is easy to see that $L_1 + L_2 = L(G)$.

We now introduce a reduced-order transformation $y = Ex$, where E is defined as above, $y^T = [y_1, \cdots, y_{n-1}]$. Note that $x = x_1\tilde{1} - Fy$ and $L_1\tilde{1} = L_2\tilde{1} = 0$. Then, system (4) reduces to the following equation

$$\dot{y}(t) = E\dot{x}(t) = EL_1Fy(t) + EL_2Fy(t-\eta)$$
$$+EDF[y(t-\tau) - y(t-\eta)] + ED[x_1(t-\eta) - x_1(t-\tau)]\tilde{1}. \tag{5}$$

On the other hand,

$$x_1(t-\eta) - x_1(t-\tau) = -\int_{t-\eta}^{t-\tau} \dot{x}_1(s)ds = \alpha\int_{t-\eta}^{t-\tau} y(s)ds \tag{6}$$

where $\alpha = [a_{12}, a_{13}, \cdots, a_{1n}]$. Thus, (5) and (6) yield

$$\dot{y}(t) = \tilde{L}_1 y(t) + \tilde{D}_1 y(t-\tau) + (\tilde{L}_2 - \tilde{D}_1)y(t-\eta) + \tilde{D}_2\int_{t-\eta}^{t-\tau} y(s)ds \tag{7}$$

where

$$\tilde{L}_1 = EL_1F, \tilde{L}_2 = EL_2F,$$
$$\tilde{D}_1 = EDF, \tilde{D}_2 = ED\Phi,$$
$$\Phi^T = [\alpha^T, \cdots, \alpha^T]_{(n-1)\times n}.$$

Based on the definition of y, the following lemma is immediate.

Lemma 2. If system (7) is asymptotically stable, then, system (4) asymptotically solves consensus.

Lemma 3. If $G = (V, E, A)$ has a spanning tree, then, there exist matrices $P > 0, Q_i \geq 0$, $i = 1, 2, \cdots, 5$, and positive constants σ, τ such that the following linear matrix inequality holds:

$$\Psi = [\Omega_{ij}] + \Xi^T(\tau Q_3 + \sigma Q_4)\Xi < 0, \tag{8}$$

where

$$\Omega_{11} = P\tilde{L}_1 + \tilde{L}_1^T P + Q_1 - \tau^{-1}Q_3 + \sigma Q_5,$$
$$\Omega_{12} = P\tilde{D}_1 + \tau^{-1}Q_3, \Omega_{13} = P(\tilde{L}_2 - \tilde{D}_1),$$
$$\Omega_{14} = P\tilde{D}_2,$$
$$\Omega_{22} = -Q_1 + Q_2 - \tau^{-1}Q_3 - \sigma^{-1}Q_4,$$
$$\Omega_{23} = \sigma^{-1}Q_4, \Omega_{24} = \Omega_{34} = 0,$$
$$\Omega_{33} = -Q_2 - \sigma^{-1}Q_4, \Omega_{44} = -\sigma^{-1}Q_5,$$
$$\Xi^T = [\tilde{L}_1, \tilde{D}_1, \tilde{L}_1 - \tilde{D}_1, \tilde{D}_2].$$

Proof. By Lemma 1, there exists a positive matrix P such that

$$P\tilde{L}(G) + \tilde{L}^T(G)P < 0, \quad \tilde{L}(G) = EL(G)F.$$

Let $Q_1 = Q_2 = 0$ and denote Ω with $Q_1 = Q_2 = 0$ by Ω_0. Note that $\tilde{L}_1 + \tilde{L}_2 = \tilde{L}(G)$, we have

$$\Gamma^T\Omega_0\Gamma = \begin{bmatrix} \tilde{\Omega}_{11} & P\tilde{L}_2 & \Omega_{13} & \Omega_{14} \\ \tilde{L}_2^T P & \tilde{\Omega}_{22} & 0 & 0 \\ \Omega_{13}^T & 0 & \tilde{\Omega}_{33} & 0 \\ \Omega_{14}^T & 0 & 0 & \tilde{\Omega}_{44} \end{bmatrix},$$

where

$$\Gamma = \begin{bmatrix} I_{n-1} & 0 & 0 & 0 \\ I_{n-1} & I_{n-1} & 0 & 0 \\ 0 & I_{n-1} & I_{n-1} & 0 \\ 0 & 0 & 0 & I_{n-1} \end{bmatrix},$$

$\tilde{\Omega}_{11} = P\tilde{L}(G) + \tilde{L}^T(G)P + \sigma Q_5$, $\tilde{\Omega}_{22} = -\tau^{-1}Q_3$,

$\tilde{\Omega}_{33} = -\sigma^{-1}Q_4$, $\tilde{\Omega}_{44} = -\sigma^{-1}Q_5$.

It is not difficult to conclude that $\Gamma^T\Omega_0\Gamma < 0$ for $Q_3 = Q_4 = Q_5 = I_{n-1}$ and sufficiently small τ, σ. Therefore, if $G = (V, E, A)$ has a spanning tree, then (8) holds for $Q_1 = Q_2 = 0$, $Q_3 = Q_4 = Q_5 = I_n$ and sufficiently small $\tau, \sigma > 0$.

Lemma 4 [15]. For any continuous vector $u(t)$ on $[t-\tau, t]$ with $t \geq 0$, $\tau > 0$ and positive definite matrix $W > 0$, the following inequality holds:

$$\int_{t-\tau}^{t} u^T(s)Wu(s)ds$$
$$\geq \tau^{-1}\left(\int_{t-\tau}^{t} u(s)ds\right) W \left(\int_{t-\tau}^{t} u(s)ds\right).$$

Now, let us give the main result of this paper.

Theorem 5. If $G = (V, E, A)$ has a spanning tree, then, system (4) asymptotically solves consensus for tolerable delays $\sigma, \tau > 0$ satisfying (8).

Proof. By Lemma 2, it is sufficient to prove that system (7) is asymptotically stable. By Lemma 3, there exist matrices $P > 0, Q_i \geq 0$, $i = 1, 2, \cdots, 5$, and positive constants σ, τ, such that (8) holds. Define the following Lyapunov function

$$V(t) = y^T(t)Py(t) + \int_{t-\tau}^{t} y^T(s)Q_1y(s)ds$$
$$+\int_{t-\eta}^{t-\tau} y^T(s)Q_2y(s)ds$$
$$+\int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(s)Q_3\dot{y}(s)dsd\beta$$
$$+\int_{-\eta}^{-\tau}\int_{t+\beta}^{t} \dot{y}^T(s)Q_4\dot{y}(s)dsd\beta$$
$$+\int_{-\eta}^{-\tau}\int_{t+\beta}^{t} y^T(s)Q_5y(s)dsd\beta.$$

Along the solution of system (7), we have

$$\begin{aligned} \dot{V}(t) &= y^T(t)[P\tilde{L}_1 + \tilde{L}_1^T P + Q_1 + \sigma Q_5]y(t) \\ &+2y^T(t)P\tilde{D}_1 y(t-\tau) \\ &+2y^T(t)P(\tilde{L}_2 - \tilde{D}_1)y(t-\eta) \\ &+2y^T(t)P\tilde{D}_2\int_{t-\eta}^{t-\tau} y(s)ds \\ &+y^T(t-\tau)[Q_2 - Q_1]y(t-\tau) \\ &-y^T(t-\eta)Q_2y(t-\eta) \\ &+\dot{y}^T(t)[\tau Q_3 + \sigma Q_4]\dot{y}(t) \\ &-\int_{t-\tau}^{t} \dot{y}^T(s)Q_3\dot{y}(s)ds \\ &-\int_{t-\eta}^{t-\tau} \dot{y}^T(s)Q_4\dot{y}(s)ds \\ &-\int_{t-\eta}^{t-\tau} y^T(s)Q_5y(s)ds. \end{aligned} \tag{9}$$

By Lemma 4, we have that

$$\begin{aligned} &\int_{t-\tau}^{t} \dot{y}^T(s)Q_3\dot{y}(s)ds \geq \tau^{-1}\xi_1^T(t)Q_3\xi_1(t) \\ &\int_{t-\eta}^{t-\tau} \dot{y}^T(s)Q_4\dot{y}(s)ds \geq \sigma^{-1}\xi_2^T(t)Q_4\xi_2(t) \\ &\int_{t-\eta}^{t-\tau} y^T(s)Q_5y(s)ds \geq \sigma^{-1}\xi_3^T(t)Q_5\xi_3(t), \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\xi_1(t) = [y(t) - y(t-\tau)], \\ &\xi_2(t) = [y(t-\tau) - y(t-\eta)], \\ &\xi_3(t) = \int_{t-\eta}^{t-\tau} y(s)ds. \end{aligned} \tag{11}$$

By (8)-(11), we get

$$\dot{V}(t) = \xi^T(t)\Omega\xi(t) < 0, \quad t \geq 0,$$

where $\xi^T(t) = [y^T(t), y^T(t-\tau), y^T(t-\eta), \xi_3^T(t)]$. This implies that system (7) is asymptotically stable. Hence, system (4) asymptotically solves consensus.

Remark 1. Theorem 5 shows that consensus of system (4) is robust to delays σ and τ. In practice, for given $\sigma, \tau > 0$, we can directly verify the linear matrix inequality (8) by Matlab's LMI tool box.

Remark 2. For the case of multiple additive delays, the above method is also valid. It only brings about much computation.

4. SIMULATION RESULTS

Consider a digraph $G = (V, E, A)$ with $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,1)\}$, and $A = \begin{bmatrix} \tilde{0}^T & 1 \\ I_5 & \tilde{0} \end{bmatrix}$

Here, we assume that the first agent's dynamic equation is free of delay, $\tau = 0.25$ and $\sigma = 0.5$. Based on Matlab's LMI tool box, we find that (8) is feasible. Thus, by Theorem 5, the system asymptotically solves consensus. The state of the system with a stochastic initial state $x(0)$ is shown in Fig. 1.

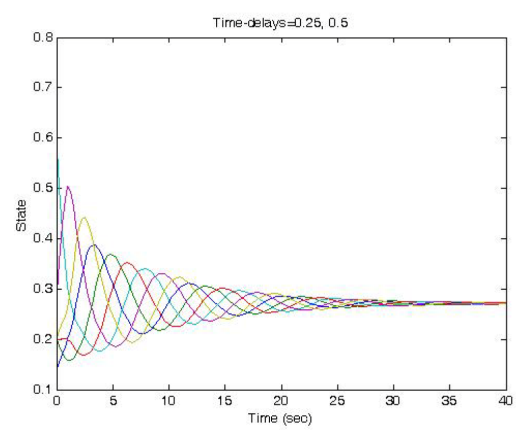

Fig. 1: State trajectories of the system for the case when the first agent's dynamic equation is free of delay.

When the first two agents' dynamic equations are free of delay, the state of the system with a stochastic initial state $x(0)$ is shown in Fig. 2.

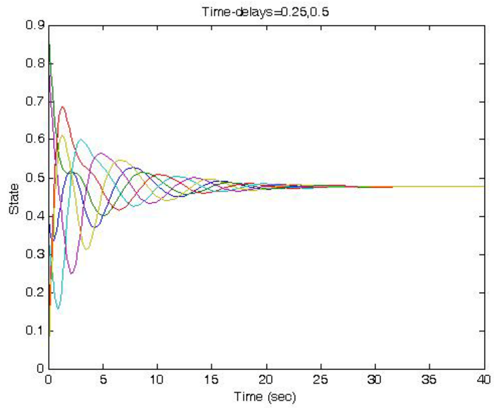

Fig. 2: State trajectories of the system for the case when the first two agents' dynamic equation is free of delay.

It is obvious that the speed of convergence to consensus increases if the number of agents whose dynamic equations are free of delay increases.

5. CONCLUSIONS

In this paper, we consider the consensus problem for continuous time multi-agent systems with additive delays. When assuming that there at least exists one agent whose dynamic equation is free of delay, By transforming the multi-agent system into a reduced-order system with distributed delays, we prove that if the digraph G has a spanning tree, then the system asymptotically solves consensus for appropriate delays. One feasible linear matrix inequality is also given to determine the upper bound of delays guaranteeing the achievement of consensus of the system.

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